Ex 1. 3 *What is the minimum value of $H(p_1, \ldots, p_n) = H(\mathbf{p})$ as $\mathbf{p}$ ranges over the set of n-dimensional probability vectors? Find all $\mathbf{p}$'s which achieve this minimum.*

$$\{(p_1, \ldots, p_n) \in [0,1]^n \mid \sum_{i=1}^n p_i = 1,\ H(p) \text{ minimal}\}$$

$$= \{(1,0,0,\ldots,0), (0,1,0,0,\ldots,0), \ldots, (0,0,\ldots,1)\}$$

$$H(p) = -\sum_{i=1}^n p_i \cdot \log_2(p_i) \overset{p=(1,0,\ldots,0)}{=}$$

$$= -\left(p_1 \cdot \log_2(p_1) + \sum_{i=2}^n p_i \log_2(p_i)\right)$$

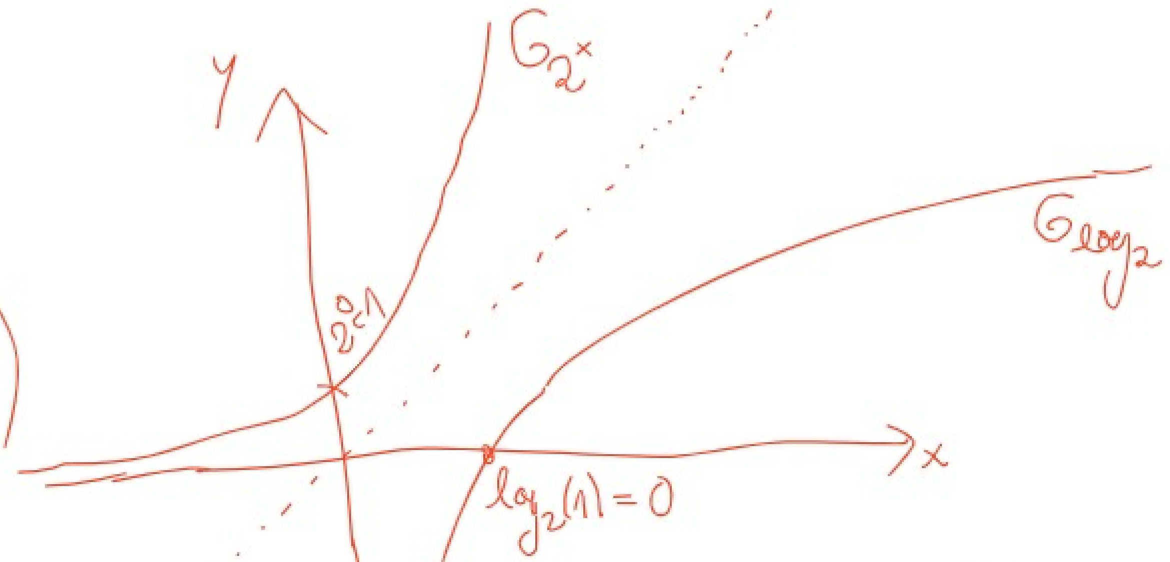
$$= -\left(1 \cdot \underbrace{\log_2(1)}_{=0} + (n-1) \cdot \underbrace{\lim_{p \downarrow 0} p \cdot \log_2(p)}_{=0}\right)$$

$$= \underline{\underline{0}}$$

$\log_2(2^x) = x$

$\log_2(1) = \log_2(2^0) = 0$

$$\lim_{x \downarrow 0} x \cdot \frac{1}{x^2} = \lim_{x \downarrow 0} \frac{1}{x} = \infty$$



$D_{\log_2} =]0, \infty[$

$\lim_{x \downarrow 0} \log_2(x) = -\infty$

$$\lim_{x \downarrow 0} x \cdot \log_2(x) = \lim_{x \downarrow 0} \frac{\log_2(x)}{\frac{1}{x}} =$$

$$= \lim_{x \downarrow 0} \frac{(\log_2(x))'}{(\frac{1}{x})'} = \lim_{x \downarrow 0} \frac{\frac{1}{x \cdot \ln(2)}}{(-1) \cdot \frac{1}{x^2}} = \lim_{x \downarrow 0} \frac{\frac{1}{\ln(2)}}{-\frac{1}{x}} = \lim_{x \downarrow 0} -\frac{x}{\ln(2)} = \underline{\underline{0}}$$

$(x^{10})' = 10x^9$

$\boxed{(x^n)' = n \cdot x^{n-1}}$

$(\frac{1}{x})' = (x^{-1})' = (-1) \cdot x^{-2}$

$\boxed{(e^x)' = e^x \quad (\ln x)' = \frac{1}{x}}$

$\boxed{(a^x)' = a^x \cdot \ln(a) \quad (\log_a(x))' = \frac{1}{x \cdot \ln(a)}}$

independent, identically distributed

X, Y independent $\Leftrightarrow$
$\forall A, B: \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$

proof +

$$\left(\frac{1}{x}\right)' = (x^{-1})' = (-1)\cdot x^{-2}$$

X, Y independent $\Leftrightarrow$
$\forall A,B: \ P(X\in A, Y\in B) = \mathbb{P}(X\in A)\cdot \mathbb{P}(Y\in B)$

independent, identically distributed

Ex 1.8 *Let $X_1, X_2, \ldots,$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let $C_n(t) = \{x^n \in X^n : p(x^n) \geq 2^{-nt}\}$ denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.*

$x^n = (x_1, x_2, \ldots, x_n) \in X^n$

a) *Show $|C_n(t)| \leq 2^{nt}$* $= 2^{nt}$

b) *For what values of t does $P(X^n \in C_n(t)) \to 1$?*

c) *Denoted by $p(x)$ the probability mass of the random variable X compute the limit*

$$\lim_{n\to\infty} (p(X_1, X_2, \ldots, X_n))^{1/n}$$

$$C_n(t) = \{(x_1, \ldots, x_n) \in X^n \mid p(x_1, \ldots, x_n) = \mathbb{P}(X_1 = x_1, \ldots, X_n = x_n) \geq 2^{-nt}\}$$

1.8 a) Assume for contradiction $|C_n(t)| > 2^{n\cdot t}$.

$$\mathbb{P}(C_n(t)) = \sum_{x^n \in C_n(t)} \underbrace{p(x^n)}_{\geq 2^{-nt}} \geq \sum_{x^n \in C_n(t)} 2^{-nt} = \underbrace{|C_n(t)|}_{> 2^{nt}} \cdot 2^{-nt} > 2^{nt}\cdot 2^{-nt} = 2^{nt-nt} = 2^0 = 1$$

$$\boxed{a^n \cdot a^m = a^{n+m}}$$

$\Rightarrow \mathbb{P}(C_n(t)) > 1$
Contradiction ↯
$\Rightarrow |C_n(t)| \leq 2^{n\cdot t}$

$$\sum_{i=3}^{7} 5 = |\{3,4,5,6,7\}| \cdot 5 = 5\cdot 5 = 25$$

X, Y independent $\Leftrightarrow$
$\forall A,B: \ P(X\in A, Y\in B) = \mathbb{P}(X\in A)\cdot \mathbb{P}(Y\in B)$

Ex 1.8 *Let $X_1, X_2, \ldots,$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let $C_n(t) = \{x^n \in X^n : p(x^n) \geq 2^{-nt}\}$ denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.*

- *Show $|C_n(t)| \leq 2^{-nt}$.*
- *For what values of t does $P(X^n \in C_n(t)) \to 1$?*
- *Denoted by $p(x)$ the probability mass of the random variable X compute the limit*

$$\lim_{n\to\infty} (p(X_1, X_2, \ldots, X_n))^{1/n}$$

$$\ln\left(\left(\mathbb{P}(X_1 = x_1, \ldots, X_n = x_n)\right)^{\frac{1}{n}} \right) \overset{X_1, \ldots, X_n \text{ independent}}{=} \ldots \mathbb{P}(X_n = x_n)\Big)^{\frac{1}{n}}\Big)$$

$$a^{n+m} = a^n \cdot a^m$$

$$\ldots = \ln(x) + \log_a(y)$$

$= 5 \cdot 5 =$

X, Y independent $\Leftrightarrow$

$\forall A, B: \quad \mathbb{P}(X \in A, Y \in B) = \mathbb{P}(X \in A) \cdot \mathbb{P}(Y \in B)$

Ex 1. 8 *Let $X_1, X_2, \dots,$ be an i.i.d. sequence of discrete random variables with entropy $H(X)$. Let $C_n(t) = \{x^n \in X^n : p(x^n) \geq 2^{-nt}\}$ denote the subset of n-sequences with probabilities $\geq 2^{-nt}$.*

- *Show $|C_n(t)| \leq 2^{-nt}$.*
- *For what values of t does $P(X^n \in C_n(t)) \to 1$?*
- *Denoted by $p(x)$ the probability mass of the random variable X compute the limit*

$$\lim_{n\to\infty} (p(X_1, X_2, \dots, X_n))^{1/n}$$

$X_1, X_2, X_3, \dots$ i.i.d., $\mathbb{E}[|X_1|] < \infty$

Strong law of large numbers

$$\frac{1}{n}\sum_{i=1}^{n} X_i \xrightarrow[\text{almost surely}]{n\to\infty} \mathbb{E}[X_1]$$

$$\ln\left(\left(\mathbb{P}(X_1 = x_1, \dots, X_n = x_n)\right)^{\frac{1}{n}}\right) = \in X_1, \dots, X_n \text{ independent}$$

$$= \ln\left(\left(\mathbb{P}(X_1 = x_1) \cdot \dots \cdot \mathbb{P}(X_n = x_n)\right)^{\frac{1}{n}}\right)$$

$$= \frac{1}{n} \cdot \ln\left(\mathbb{P}(X_1 = x_1) \cdot \dots \cdot \mathbb{P}(X_n = x_n)\right)$$

$$= \frac{1}{n} \cdot \left(\ln(\mathbb{P}(X_1 = x_1)) + \dots + \ln(\mathbb{P}(X_n = x_n))\right)$$

$a^{n+m} = a^n \cdot a^m$

$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

$= \ln(a^b) = b \cdot \ln(a)$

$\mathbb{P}_{X_1} = \mathbb{P}_X$

$p = \mathbb{P}_{(X_1, \dots, X_n)}$

$p(x_1, \dots, x_n) = \mathbb{P}_{X_1}(x_1) \cdot \dots \cdot \mathbb{P}_{X_n}(x_n)$

$= \mathbb{P}_X(x_1) \cdot \dots \cdot \mathbb{P}_X(x_n)$

$$\ln\left(p(X_1, X_2, \dots, X_n)^{\frac{1}{n}}\right)$$

$$= \frac{1}{n}\left(\ln(\mathbb{P}_X(X_1)) + \dots + \ln(\mathbb{P}_X(X_n))\right) \xrightarrow[\text{a.s. (almost surely)}]{n\to\infty} \mathbb{E}[\ln(\mathbb{P}_X(X_1))] \quad \Big| e^{(\,)}$$

$$p(X_1, \dots, X_n)^{\frac{1}{n}} \xrightarrow[\text{a.s.}]{n\to\infty} 2^{\mathbb{E}[\log_2(\mathbb{P}_X(X_1))]} = 2^{-H(X_1)}$$